DSC 40B

Theoretical Foundations II

Lecture 8 | Part 1

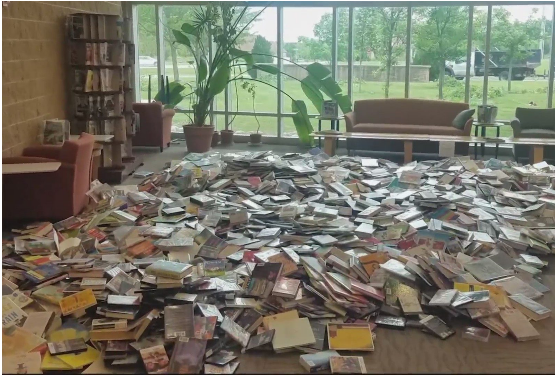
Dynamic Sets

Bookkeeping

- How do you store your books?

Bookkeeping

- How do you store your books?



Bookkeeping

- How do you store your books?

https://bookriot.com/how-to-organize-bookshelves/

Bookkeeping: Tradeoffs

- Messy:
  - No upfront cost.
  - Cost to search is high.
- Organized
  - Big upfront cost.
  - Cost to search is low.
- "Right" choice depends on how often we search.

- cost to add
  a book?

Data Structures and Algorithms

- **Data structures** are ways of organizing data to make certain operations faster.
- Come with an upfront cost (preprocessing).
- “Right” choice of data structure depends on what operations we’ll be doing in the future.

Queries: Easy to Hard

- We've been thinking about **queries**.
  - Given a collection of data, is x in the collection?
- Querying is a fundamental operation.
  - Useful in a data science sense.
  - But also frequently performed in algorithms.
- There are several situations to think about.

Situation #1: Static Set, One Query

- **Given**: an unsorted collection of n numbers (or strings, etc.).
- In future, you will be asked single query.
- Which is better: linear search or sort + binary search?

Situation #1: Static Set, One Query

- **Given**: an unsorted collection of n numbers (or strings, etc.).

- In future, you will be asked single query.

- Which is better: linear search or sort + binary search?
  - Linear search: $\Theta(n)$ worst case.
  - Binary search would require sorting first in $\Theta(n \log n)$ worst case

Situation #2: Static Set, Many Queries

- **Given**: an unsorted collection of *n* numbers (or strings, etc.).
- In future, you will be asked **many** queries.
- Which is better: linear search or sort + binary search?
  - Depends on number of queries!

Exercise

Suppose you have a static set of n items. How long will it take[a] to perform k queries in total with:

1. linear search? $\Theta(k \cdot n)$
2. sort + binary search? $\Theta(n \log n + k \cdot \log n)$

If $k = n/10$, which should you use? $\rightarrow$ $\begin{cases} 1. & \Theta(\frac{n}{10} \cdot n) = \Theta(n^2) \\ 2. & \Theta(n \log n + \frac{n}{10} \cdot \log n) \end{cases}$

$= \Theta(n \log n)$ ✓

What if $k = \log n$?

$\begin{cases} 1. & \Theta(n \cdot \log n) \\ 2. & \Theta(n \log n + (\log n)^2) = \Theta(n \log n) \end{cases}$

---

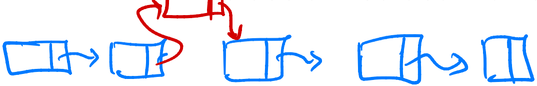
[a] On average. Assume the best case is rare.

Situation #3: Dynamic Set, Many Queries

- **Given**: a collection of n numbers (or strings, etc.).
- In future, you will be asked **many** queries *and* to **insert** new elements.
- Best approach: ?

Binary Search?

- Can we still use binary search?
- **Problem**: To us binary search, we must maintain array in sorted order as we insert new elements.
- Inserting into array takes $\Theta(n)$ time in worst case.
  - Must "make room" for new element.
  - Can we use linked list with binary search?



Exercise

Suppose we have a collection of n elements. We make $n/4$ insertions and $n/4$ queries. How long will this take in total with

1. append to linked list append + linear search?
2. maintain sorted array + binary search?

1. $\Theta(1) \cdot n/4 + \Theta(n) \cdot n/4 = \Theta(n) + \Theta(n^2) = \Theta(n^2)$

2. $\Theta(n) \cdot n/4 + \Theta(\log n) \cdot n/4 = \Theta(n^2) + \Theta(n \log n) = \Theta(n^2)$.

Today

- Introduce (or review) **binary search trees**.
- BSTs support fast queries *and* insertions.
- Preserve sorted order of data after insertion.
- Can be modified to solve many problems efficiently.
  - Example: finding order statistics.

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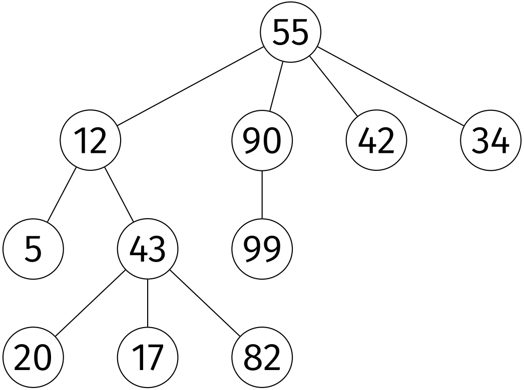
Lecture 8 | Part 2

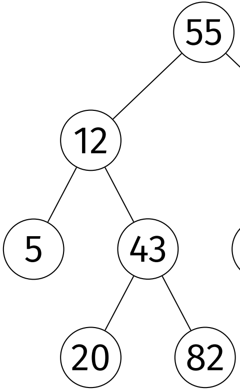
Binary Search Trees

Trees



Binary Trees

- Each node has *at most* two children (left and right).



Binary Search Tree

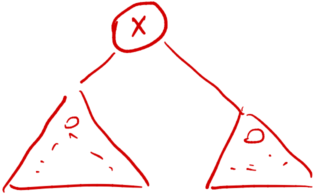
- A **binary search tree** (BST) is a binary tree that satisfies the following for any node x:

- if y is in *x*'s **left** subtree:

$$y.\text{key} \leq x.\text{key}$$

- if y is in *x*'s **right** subtree:

$$y.\text{key} \geq x.\text{key}$$



Assumption (for simplicity)

- We'll assume keys are unique (no duplicates).
- if y is in *x*'s **left** subtree:

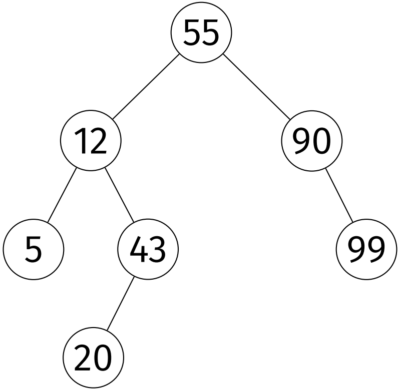
$$\texttt{y.key} < \texttt{x.key}$$

- if y is in *x*'s **right** subtree:

$$\texttt{y.key} > \texttt{x.key}$$

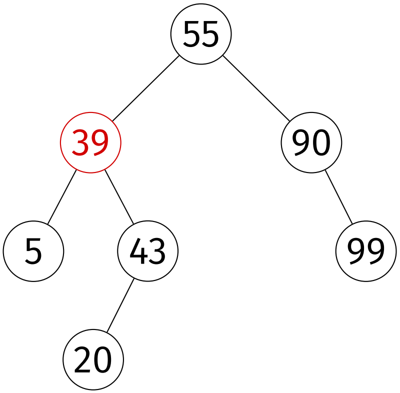
Example

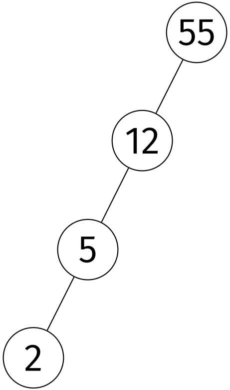
- This **is** a BST.



Example

- This is **not** a BST.

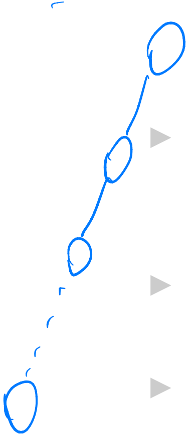




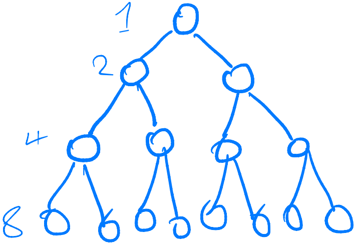
Exercise

Is this is a BST? yes

Height



- The **height** of a tree is the number of edges from the root to any leaf.
- Suppose a binary tree has n nodes.
- The **tallest** it can be is $\approx n$
- The **shortest** it can be is $\approx \log_2 n$

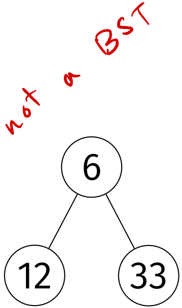


In Python

```python
class Node:
    def __init__(self, key, parent=None):
        self.key = key
        self.parent = parent
        self.left = None
        self.right = None

class BinarySearchTree:
    def __init__(self, root: Node):
        self.root = root
```

In Python




```
root = Node(6)
n1 = Node(12, parent=root)
root.left = n1
n2 = Node(33, parent=root)
root.right = n2
tree = BinarySearchTree(root)
```

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Theoretical Foundations II

Lecture 8 | Part 3

Queries and Insertions in BSTs

Why?

- BSTs impose structure on data.
- “Not quite sorted”.
- Preprocessing for making insertions *and* queries faster.

Operations on BSTs

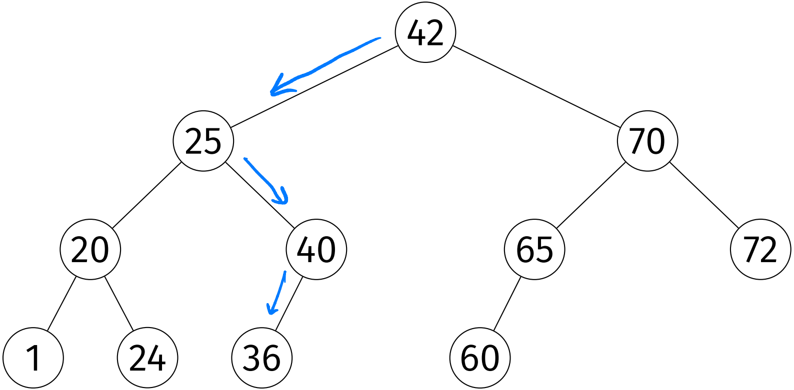
- We will want to:
  - **query** a key (is it in the tree?)
  - **insert** a new key

Queries

- **Given**: a BST and a target, t.
- **Return**: `True` or `False`, is the target in the collection?

Queries

- Is 36 in the tree? 65? 23?



Queries

- Start walking from root.
- If current node is:
  - equal to target, return `True`;
  - too large (> target), follow left edge;
  - too small (< target), follow right edge;
  - `None`, return `False`

Queries, in Python

```
def query(self, target):
    """As method of BinarySearchTree."""
    current_node = self.root
    while current_node is not None:
        if current_node.key == target:
            return current_node
        elif current_node.key < target:
            current_node = current_node.right
        else:
            current_node = current_node.left
    return None
```

Exercise

Complete the recursive version of `query`.

```
def query_recursive(node, target):
    """As a 'free function'."""
    if node is None:
        return False

    if node.key == target:
        ...
    elif ...:
        ...
    else:
        ...
```

Queries (Recursive)

```python
def query_recursive(node, target):
    """As a 'free function'."""
    if node is None:
        return False

    if node.key == target:
        return node
    elif node.key < target:
        return query_recursive(node.right, target)
    else:
        return query_recursive(node.left, target)
```

Queries, Analyzed

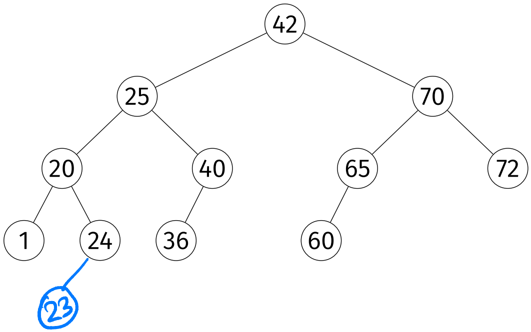
- Best case: $\Theta(1)$.
- Worst case: $\Theta(h)$, where h is **height** of tree.

Insertion

- **Given**: a BST and a new key, k.
- **Modify**: the BST, inserting k.
- Must **maintain** the BST properties.

Insertion

- Insert 23 into the BST.



Insertion (The Idea)

- Traverse the tree as in `query` to find empty spot where new key should go, keeping track of last node seen.
- Create new node; make last node seen the parent, update parent's children.
- Be careful about inserting into empty tree!

```python
def insert(self, new_key):
    # assume new_key is unique
    current_node = self.root
    parent = None
```

$\Theta(h)$

```python
    # find place to insert the new node
    while current_node is not None:
        parent = current_node
        if current_node.key < new_key:
            current_node = current_node.right
        else: # current_node.key > new_key
            current_node = current_node.left
```

$\Theta(h)$

$\Theta(1)$

```python
    # create the new node
    new_node = Node(key=new_key, parent=parent)

    # if parent is None, this is the root. Otherwise, update the
    # parent's left or right child as appropriate
    if parent is None:
        self.root = new_node
    elif parent.key < new_key:
        parent.right = new_node
    else:
        parent.left = new_node
```

Insertion, Analyzed

- Worst case: $\Theta(h)$, where h is **height** of tree.

Main Idea

Querying and insertion take $\Theta(h)$ time in the worst case, where h is the height of the tree.

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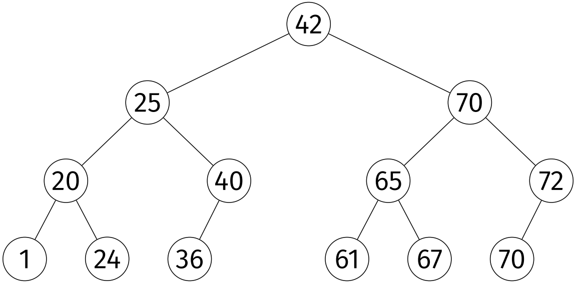
Lecture 8 | Part 4

Balanced and Unbalanced BSTs

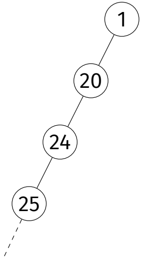
Binary Tree Height

- In case of very balanced tree, $h = \Theta(\log n)$.
  - Query, insertion take worst case $\Theta(\log n)$ time in a **balanced** tree.



Binary Tree Height

- In the case of very unbalanced tree, $h = \Theta(n)$.
  - Query, insertion take worst case $\Theta(n)$ time in **unbalanced** trees.

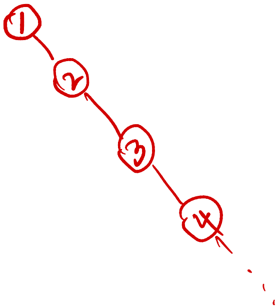


Unbalanced Trees

- Occurs if we insert items in (close to) sorted or reverse sorted order.
- This is a **common** situation.

Example

- Insert ~~1~~, ~~2~~, ~~3~~, 4, 5, 6, 7, 8 (in that order).



Time Complexities

query	$\Theta(h)$
insertion	$\Theta(h)$

Where h is height, and $h = \Omega(\log n)$ and $h = O(n)$.

Time Complexities (Balanced)

query	$O(\log n)$
insertion	$O(\log n)$

Where h is height, and $h = \Omega(\log n)$ and $h = O(n)$.

Worst Case Time Complexities (Unbalanced)

query	$\Theta(n)$
insertion	$\Theta(n)$

- The worst case is **bad**.
  - Worse than using a sorted array!
- The worst case is **not rare**.

Main Idea

The operations take linear time in the worst case **unless** we can somehow ensure that the tree is **balanced**.

Self-Balancing Trees

- There are variants of BSTs that are **self-balancing**.
  - Red-Black Trees, AVL Trees, etc.
- Quite complicated to implement correctly.
- But their height is **guaranteed** to be $\sim \log n$.
- So insertion, query take $\Theta(\log n)$ in worst case.

Warning!

If asked for the time complexity of a BST operation, be careful! A common mistake is to say that insertion/query are $\Theta(\log n)$ without being told that the tree is balanced.

Main Idea

In general, insertion/query take $\Theta(h)$ time in worst case. If tree is balanced, $h = \Theta(\log n)$, so they take $\Theta(\log n)$ time. If tree is badly unbalanced, $h = O(n)$, and they can take $O(n)$ time.

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Theoretical Foundations II

Lecture 8 | Part 5

Augmenting BSTs

Modifying BSTs

- Perhaps more than most other data structures, BSTs must be modified (**augmented**) to solve unique problems.

Order Statistics

- Given n numbers, the **kth order statistic** is the kth smallest number in the collection.

Example

```
[99, 42, -77, -12, 101]
```

- 1st order statistic: -77
- 2nd order statistic: -12
- 4th order statistic: 99

Dynamic Set, Many Order Statistics

- Quickselect finds any order statistic in linear expected time.

- This is efficient for a static set.

- Inefficient if set is dynamic.

Goal

- Create a **dynamic** set data structure that supports fast computation of **any** order statistic.

BST Solution

- For each node, keep attribute `.size`, containing # of nodes in subtree rooted at current node

- Property:[1]
  `x.size = x.left.size + x.right.size + 1`

x

x.size = 1

[1] If a left or right child doesn't exist, consider its size zero.

Computing Sizes

```
def add_sizes_to_tree(node):
    if node is None:
        return 0
    left_size = add_sizes_to_tree(node.left)
    right_size = add_sizes_to_tree(node.right)
    node.size = left_size + right_size + 1
    return node.size
```

$\Theta(n)$

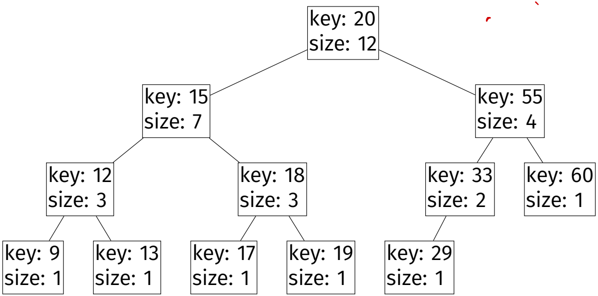
Note

- Also need to maintain size upon inserting a node.

Computing Order Statistics

- 8th? 2nd? 12th

kth-OS(root, k)



Augmenting Data Structures

- This is just one example, but many more.
- Understanding how BSTs work is key to augmenting them.