

# DSC 40B

## *Theoretical Foundations II*

Lecture 12 | Part 1

**Warmup: Aggregate Analysis**

# Time Complexity

```
def full_bfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered':
            bfs(graph, node, status)

def bfs(graph, source, status=None):
    """Start a BFS at `source`."""
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

$\theta(V)$

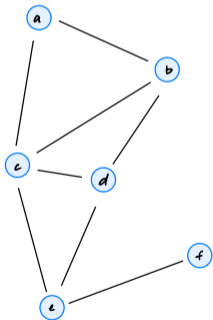
→

$\theta(E)$

Total :  $\theta(V + E)$

## Exercise

What is printed if we run a BFS starting at a?



```
...  
while pending:  
    u = pending.popleft()  
    print(f'Popped {u}')  
    for v in graph.neighbors(u):  
        print(f'Exploring edge ({u}, {v})')  
        # explore edge (u,v)  
    ...
```

# Answer

Popping a

Exploring edge (a, b)

Exploring edge (a, c)

Popping b

Exploring edge (b, a)

Exploring edge (b, c)

Exploring edge (b, d)

Popping c

Exploring edge (c, a)

Exploring edge (c, b)

Exploring edge (c, d)

Exploring edge (c, e)

Popping d

Exploring edge (d, b)

Exploring edge (d, c)

Exploring edge (d, e)

Popping e

Exploring edge (e, c)

Exploring edge (e, d)

Exploring edge (e, f)

Popping f

Exploring edge (f, e)

# Aggregate Analysis

- ▶ During any one call to bfs:
  - ▶ Number of printed nodes: ?
  - ▶ Number of printed edges: ?
- ▶ In **aggregate** (over all calls):
  - ▶ Number of printed nodes: *exactly*  $|V|$
  - ▶ Number of printed edges: *exactly*  $2|E|$

# Time Complexity

- ▶ Full BFS takes  $\Theta(V + E)$

# Time Complexity

- ▶ Full BFS takes  $\Theta(V + E)$
- ▶ Why not just  $\Theta(E)$ ?
- ▶  $\Theta(V + E)$  works *for all graphs*.
  - ▶ If we know more about the number of edges, we might be able to simplify.
  - ▶ E.g., if the graph is **complete**,  $E = \Theta(V^2)$ , so time complexity is  $\Theta(V + V^2) = \Theta(V^2)$ .

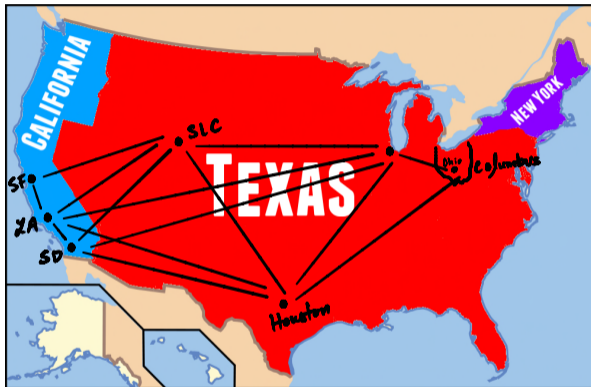
# DSC 40B

## *Theoretical Foundations II*

Lecture 12 | Part 2

**Shortest Paths**

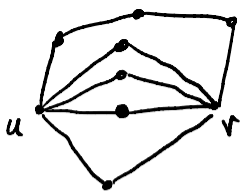




# Recall

- ▶ The **length** of a path is

$$(\# \text{ of nodes}) - 1$$



## Definitions

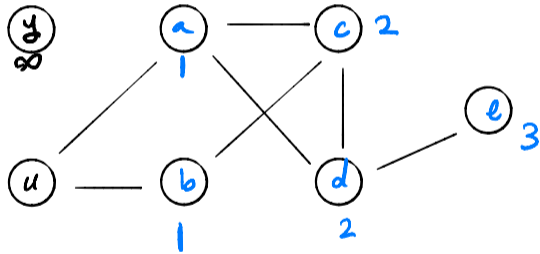


- ▶ A **shortest path** between  $u$  and  $v$  is a path between  $u$  and  $v$  with smallest possible length.
  - ▶ There may be several, or none at all.
- ▶ The **shortest path distance** is the length of a shortest path.
  - ▶ Convention:  $\infty$  if no path exists.
  - ▶ “the distance between  $u$  and  $v$ ” means spd.

# Today: Shortest Paths

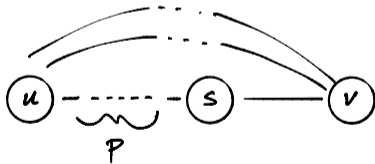
- ▶ **Given:** directed/undirected graph  $G$ , source  $u$
- ▶ **Goal:** find shortest path from  $u$  to every other node

# Example



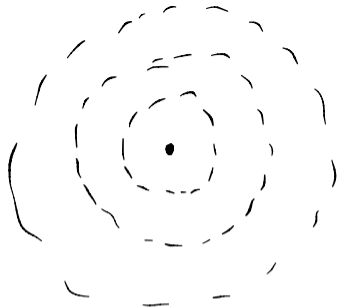
# Key Property

- ▶ A shortest path of length  $k$  is composed of:
  - ▶ A **shortest path** of length  $k - 1$ .
  - ▶ Plus one edge.



## Algorithm Idea

- ▶ Find all nodes distance 1 from source.
- ▶ Use these to find all nodes distance 2 from source.
- ▶ Use these to find all nodes distance 3 from source.
- ▶ ...



**It turns out...**

...this is exactly what BFS does.



# DSC 40B

## *Theoretical Foundations II*

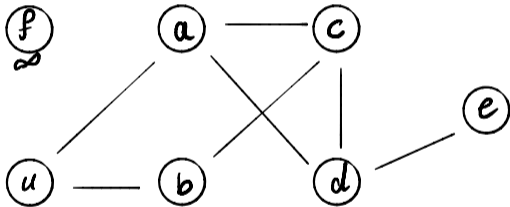
Lecture 12 | Part 3

**BFS for Shortest Paths**

# Key Property of BFS

- ▶ For any  $k \geq 1$  you choose:
- ▶ All nodes distance  $k - 1$  from source are added to the queue before any node of distance  $k$ .
- ▶ Therefore, nodes are “processed” (popped from queue) in order of distance from source.

# Example

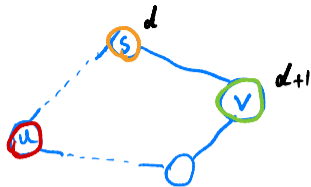


$g:$   ~~$u$~~   ~~$a$~~   ~~$b$~~   ~~$c$~~   ~~$d$~~   $e$   
0 1 1 2 2 3

# Discovering Shortest Paths

- ▶ We “discover” shortest paths when we pop a node from queue and look at its neighbors.
- ▶ But the neighbor’s status matters!

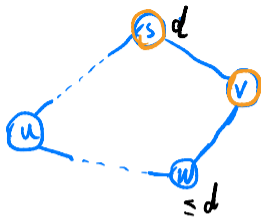
## Consider This



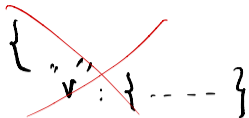
- ▶ We pop a node  $s$ .
- ▶ It has a neighbor  $v$  whose status is **undiscovered**.
- ▶ We've discovered a **shortest path** to  $v$  through  $s$ !

## Consider This

- ▶ We pop a node  $s$ .
- ▶ It has a neighbor  $v$  whose status is **pending** or **visited**.
- ▶ We already have a shortest path to  $v$ .



# Modifying BFS



- ▶ Use BFS “framework”.
- ▶ Return dictionary of **search predecessors**.
  - ▶ If  $v$  is discovered while visiting  $u$ , we say that  $u$  is the **BFS predecessor** of  $v$ .
  - ▶ This encodes the shortest paths.
- ▶ Also return dictionary of shortest path distances.

```

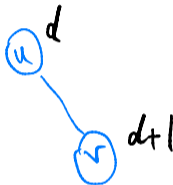
def bfs_shortest_paths(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}
    distance = {node: float('inf') for node in graph.nodes}
    predecessor = {node: None for node in graph.nodes}

    status[source] = 'pending'
    distance[source] = 0
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                distance[v] = distance[u] + 1
                predecessor[v] = u
                # append to right
                pending.append(v)
        status[u] = 'visited'

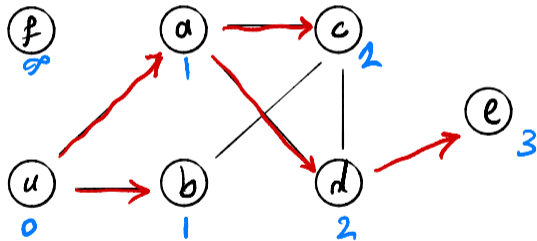
    return predecessor, distance

```





# Example



# DSC 40B

## *Theoretical Foundations II*

Lecture 12 | Part 4

**BFS Trees**

## Result of BFS

- ▶ Each node reachable from source has a single BFS predecessor.
  - ▶ Except for the source itself.
- ▶ The result is a **tree** (or forest).

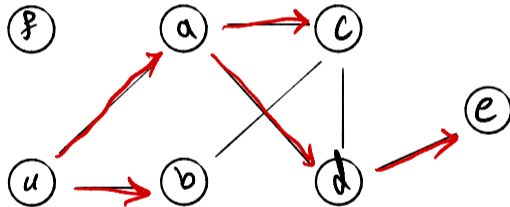


- ▶ A (free) **tree** is an undirected graph  $T = (V, E)$  such that  $T$  is connected and  $|E| = |V| - 1$ .
- ▶ A **forest** is graph in which each connected component is a tree.

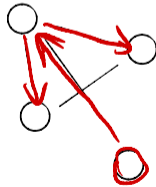
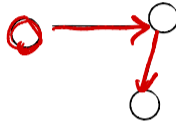
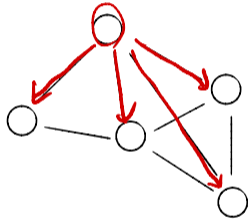
## BFS Trees (Forests)

- ▶ If the input is connected, BFS produces a **tree**.
- ▶ If the input is not connected, BFS produces a **forest**.

# Example



# Example



# BFS Trees

- ▶ BFS trees and forests encode shortest path distances.